Scottish silver arm-rings: an analysis of weights

by R Warner

This paper is an attempt to analyse the weight distribution of the Scottish silver arm-rings described by Graham-Campbell (1976). Two problems present themselves. Does the distribution of weights represent attempts to obtain one or more 'target', or preferred, values? If so, are these target-values multiples of a 'unit' (quantum)? Two methods of analysis have been used which, although capable of refinement, give results which may be found to be of interest and to warrant further discussion and study. It must be stressed that by 'target' *weight* we do not imply that actual weight was the target criterion, but simply that whatever the target criterion was it might be linearly reflected in the weight. The complete Scottish and Manx armlets have been taken together for this study, giving a total of 72 examples. (The Irish armlets will be treated elsewhere.)

A. *Fitting hypothetical distributions*

In this analysis the observed distribution of the weights was tested against various hypothetical distributions, after the distributions had been converted into frequency bar-histograms with 5 gm bar intervals (for example fig 2, below, where f_0 is the observed and $E(f)$ the expected frequency). The parameters of the expected distribution were derived from the observed data with the exception of the three isolated weights above 90 gm. The observed and expected frequency histograms were compared using both the 'Chi-square' test (in which it is sometimes necessary to combine intervals) and the 'Coefficient of Determination' (*C.D.*, the square of the 'product-moment correlation coefficient' r). The weights in a target hypothesis were assumed to be observations of a normally distributed random variable X with expectation of X, $E(X)$, equal to the target value, the mean and standard deviation of X being estimable from the observed data. In other words, it was assumed that the distribution of the weights results from normally distributed errors in the attempts to obtain the target value. Where there are two or more postulated sub-populations the total expected distribution is clearly the sum of the individual expected distributions.

A series of Null-Hypotheses is proposed, ($H_0(i)$):

$H_0(0)$. '*That the weights are rectangularly distributed.*' In other words there are *no* preferred values or targets. The limits are taken as 10 and 80 gm (in order to keep $E(f)$ to about 5). Thus $E(f)$ is constant at 4·93, and Chi^2 is found to be 41·7 with 12 degrees of freedom. This value is likely to arise, or be exceeded, in less than 0·1 % of cases. The null-hypothesis is therefore rejected, and we may infer that preferred values exist.

$H_0(1)$. '*That the observed population is homogeneous and that the recorded weights are observations of a single normally distributed random variable X.*' That is, there is only one target value. In this and all subsequent tests the limits of weight are 0 and 90 gm. The data has a mean of 43·3 gm and standard deviation of 20·6 gm. When the bar-histogram corresponding to the population of these dimensions is compared with the observed histogram we obtain a Chi^2 of 31·4 with 9 degrees of freedom, again corresponding to a probability of less than 0·001. $H_o(1)$ is therefore also rejected. (C.D. is 0·53.)

$H_o(2)$. *'That the population consists of two sub-populations, and the weight of an item from the ith population is an observation of a normally distributed random variable* X_i, $i = 1, 2$.' We would expect that the two assumed targets should be close to the two observed major modes at about 22 gm and about 48 gm, and the point of intersection should lie between 25 and 45 gm. In order to obtain the dimensions of the sub-populations we must choose this point of intersection, and because we possess no evidence to assign an individual weight near this intersect or boundary to one, or the other sub-population, we must assign any weight to that sub-population on the same side of the chosen boundary as the weight lies. If we arbitrarily place the boundary between the consecutive weights 32·47 and 34·29 gm, we obtain the variates $22{\cdot}37 \pm 4{\cdot}83$ gm (24 weights) and $50{\cdot}76 \pm 10{\cdot}37$ gm (45 weights). We obtain a Chi^2 of 9·3 with 4 degrees of freedom, representing a probability of 5%. There is therefore no reason to reject $H_o(2)$, but because of the results of $H_o(3)$ (below) further analysis of $H_o(2)$ is not undertaken. It should here be pointed out that by increasing the number of sub-populations we will always get a better fit, but Occam's razor rightly insists that, unless there are overwhelming reasons against, the simplest result which is consistent with the data should be accepted. (For $H_o(2)$ C.D. = 0·63.)

$H_o(3)$. *'That the population consists of three sub-populations, and the weight of an item from the ith sub-population is an observation of a normally distributed random variable* X_i, $i = 1, 2, 3$.' By inspection of the observed data we would expect the three postulated targets to be represented by the modes at about 22 gm, about 48 gm and about 72 gm. It would also seem not unreasonable to take the representatives of the upper (X_3) to be the 7 weights lying between 65 and 80 gm, all below 60 gm belonging to the other two. X_3 may then be described by the values $71{\cdot}7 \pm 4{\cdot}0$ gm (7 members).

The boundary difficulty between X_1 and X_2 has been mentioned above and is discussed at more length now. Although, under the assumptions made, the range of the random variables X_1 and X_2 overlap, there is no criterion for assigning a weight to one or the other sub-population. It is therefore necessary to pretend, initially, that they do not overlap and to assign rigidly and simply each side of the chosen boundary. The boundary is taken, in turn, in each interval between the 12 consecutive weights lying between 25·82 and 41·34 gm. The weights are thus assigned to X_1 and X_2 in 11 different ways, giving 11 'expected' histograms $H_o(3, 1)$ to $H_o(3, 11)$. It is found that the 'best' of these (on the Chi^2 test) would be rejected at the 5% level, but interval combination for the test is now heavy, leading to loss of detail, and the test is not thought to be appropriate. The C.D.s on the other hand (see below) are respectably high.

On fig 1 is shown, at the base of the figure, the 12 consecutive weights, and the numbered intervals, to each of which belong a histogram, a set of three sub-populations and a value for the C.D. Fig 1a shows the means of the first sub-population, m_1, and their single standard errors. Fig 1b shows the same for the second sub-population. Fig 1c shows the coefficients of determination peaking at $H_o(3, 8)$ (with a value of 0·86, although there is very little to choose between the C.D.s for $H_o(3, 6$ to $8)$. We would not be justified in choosing $H_o(3)$ rather than $H_o(2)$ (for we can make the C.D. as close to unity as we like by increasing the number of postulated sub-populations), nor in choosing $H_o(3, 8)$ in particular, were it not for the following.

On fig 1d is shown the ratio of m_3/m_1 for each boundary choice, and on fig 1e that of m_2/m_1. At $H_o(3, 8)$ (where the C.D. maximises) the value of m_3/m_1 is almost exactly 3 (3·006) and the value of m_2/m_1 almost exactly 2 (2·007). This seems to be in fair agreement with the hypothesis

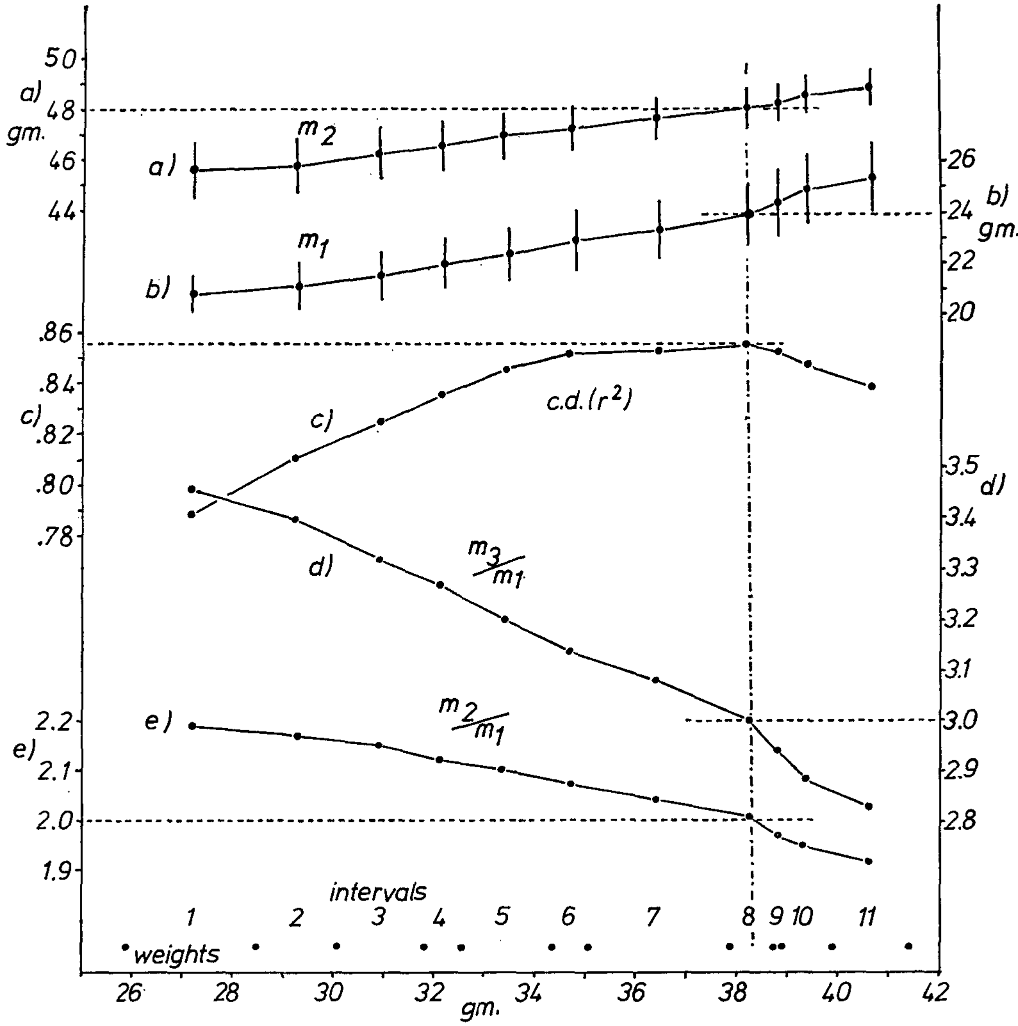

FIG 1

that the preferred values are quantic (multiples of some unit weight). $H_o(3, 8)$ seems to represent the best division of the data, and with this division $X_1 = 23{\cdot}87 \pm 6{\cdot}20$ gm (27 members), $X_2 = 47{\cdot}87 \pm 4{\cdot}40$ gm (35 members) and $X_3 = 71{\cdot}69 \pm 4{\cdot}02$ gm (7 members). The targets are therefore:

$$m_1 = 23{\cdot}87 \pm 1{\cdot}22 \text{ gm}$$
$$m_2 = 47{\cdot}87 \pm 0{\cdot}75 \text{ gm}$$
$$m_3 = 71{\cdot}69 \pm 1{\cdot}64 \text{ gm}$$

and we obtain the 'unit' values

$$m_1 = 23{\cdot}87 \pm 1{\cdot}22 \text{ gm}$$
$$\tfrac{1}{2}m_2 = 23{\cdot}89 \pm 0{\cdot}38 \text{ gm}$$
$$\tfrac{1}{3}m_3 = 23{\cdot}90 \pm 0{\cdot}55 \text{ gm}$$

which are seen to be in remarkable agreement. The 'unit' obtained by combination of these three values is $23{\cdot}88 \pm 0{\cdot}52$ gm. It will be of interest that although the standard deviations of each sub-population are high compared with the distances of the means from their boundaries, there is still only a less than 2% probability of mis-assignment. It is unlikely therefore that more than one or two weights have been mis-assigned. Fig 2 shows the expected and observed frequency histograms for $H_o(3, 8)$.

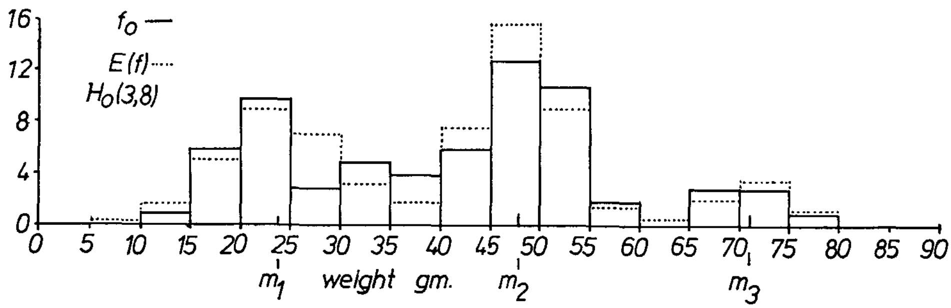

FIG 2 Frequency bar-histograms of weights in range 0–90 gm ((f_o – observed frequency; $E(f)$ – frequency expected on hypothesis $H_o(3, 8)$)

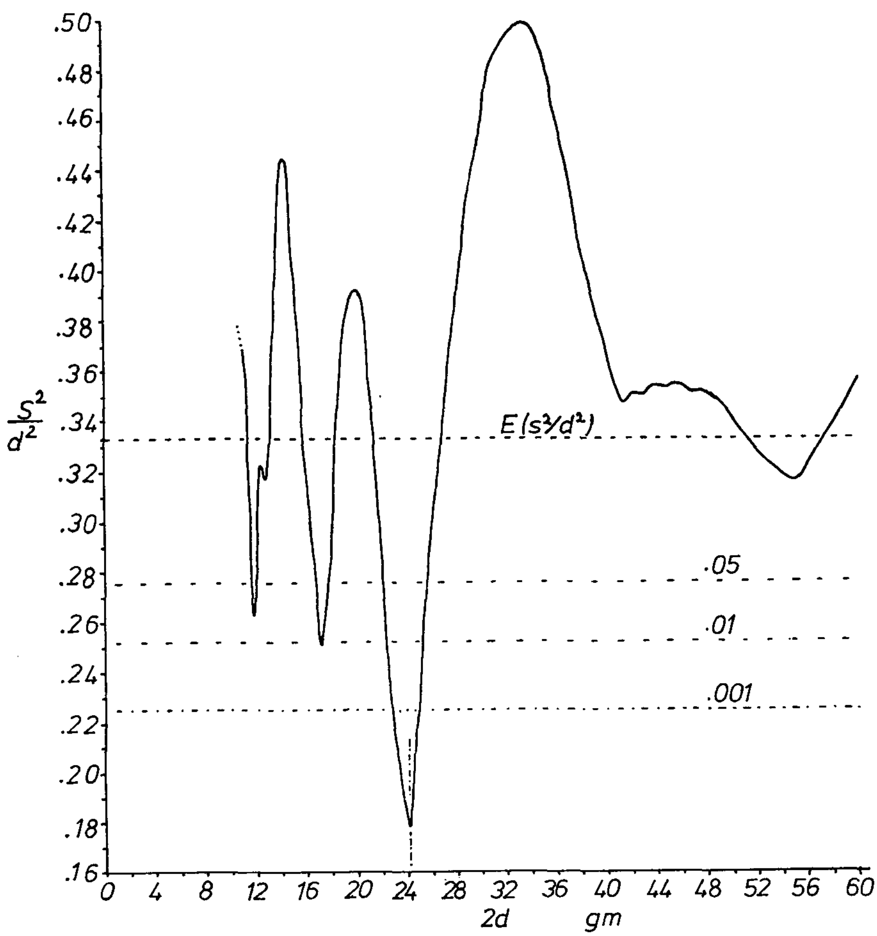

FIG 3 Behaviour of 's^2/d^2' over range of $2d$ = 11 to 60 gm

B. *Fitting hypothetical quanta*

In this analysis the data was tested for a quantic nature. The quantic hypothesis would be that '*the population consists of an indefinite number of sub-populations, and the weight of an item from the ith population is an observation of a normally distributed random variable X_i whose population mean m_i is an integral multiple of a quantum 2d*'. The search method of Broadbent (1956) was followed and the behaviour of his statistic 's^2/d^2' explored. Basically the method is a modified least-sum-of-squares approach in which 's^2/d^2' is the standardised lumped-variance of the observations about the quanta-multiples. Broadbent's modification allows the minimum values of 's^2/d^2' to be quickly obtained within the desired trial range of $2d$. If 's^2/d^2' minimises at a level significantly less than would be expected on a rectangular (non-quantic) hypothesis the data

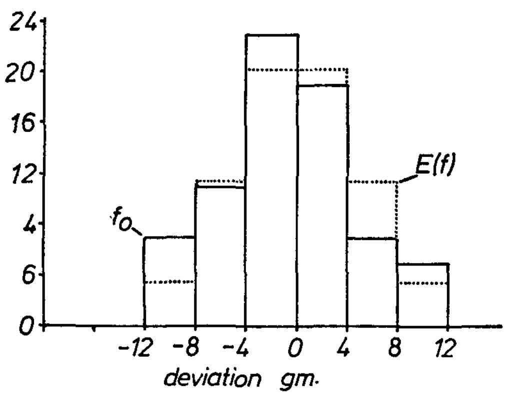

FIG 4 Frequency bar-histograms (f_o – frequency of deviation of observed weights from their nearest levels of the quantum $2d = 24{\cdot}14$ gm; $E(f)$ – expected deviation assuming normal distribution of the variable X about the quantum levels $n.2d$ ($2d = 24{\cdot}14$, $n = 1, 2 \ldots$). See section B)

may be taken as quantic and the corresponding value of $2d$ may be taken as the quantum (unit). The null hypothesis is, therefore, that the distribution is 'rectangular', and the behaviour of 's^2/d^2' tested against this. The theory is fully discussed by Broadbent (1955; 1956).

Fig 3 shows the change of 's^2/d^2' with $2d$ over the range of $2d = 11$ to 60 gm. All the weights were used in this analysis. The mean value of 's^2/d^2' expected on the rectangular hypothesis, $E(s^2/d^2)$, is shown, as are the levels of probability of the observed values deriving from that hypothesis.

It will be seen that the curve only falls below the 1% level once, at $2d = 24{\cdot}144$ gm. The probability for this minimum level is found (using the assymptotic approximation to the normal error function) to be 0·00001 (0·001%). This strongly suggests a quantum value of 24·14 gm. Because of the assignment problems discussed in A above, Broadbent's table 2 must be used to estimate the standard deviation of the observations about the quantum levels. This is 5·3 gm, giving a value for the quantum of $24{\cdot}14 \pm 0{\cdot}62$ gm. Assignment is strictly on the basis of *nearest* quantum level (integral multiple of $2d$). The sub-population boundaries are therefore midway between the quantum levels. This is different from the assignment rule in section A. On fig 4 are shown the observed and expected histograms of the deviation of the weights from their nearest level of $2d = 24{\cdot}14$.

We may list the probable number of armlets for each unit multiple thus:

	method A (23·9 gm)	method B (24·1 gm)
1 unit	27	26
2 units	35	36
3 units	7	7
4 units	2	2
5 units	1	1

CONCLUSIONS

The values of the unit found in A and B above, 23·9 ± 0·5 gm and 24·1 ± 0·6 gm, are statistically indistinguishable. Although either value can be taken as the unit it is convenient to combine our results to obtain the estimate 24·0 ± 0·8 gm. We are quite justified in concluding that the manufacturers of the arm-rings were aiming at this target, although the standard deviation of the production, 5 gm, suggests that they were not too careful about their accuracy. It is, of course, not impossible that one or more extra preferred values are also present (for example around 35 gm), but we are not justified in pursuing such a possibility on the data available. The 24 gm unit which we have found is not surprising. Skaare (1976, 50, table 13) lists bronze and lead balance-weights of 'Viking' date from Norway. Neither Skaare nor Brøgger (whom the former quotes) has undertaken a statistical analysis of the weights along the lines suggested in this paper, but preliminary inspection supports their claim that the weights represent multiples (and sub-multiples) of the *øre*, a unit of about 24 gm. Indeed, the 21 examples which could be said to represent 1 *øre* give a *mean* of 24·1 ± 0·1 gm, and the multiples give a very similar value. We can hardly doubt that our unit is also the *øre*. We also find (Skaare, *ibid*) that silver 'rings' from four Scandinavian hoards peak at what appear to be *øre* multiples, and give a unit of 24·4 ± 1 gm (provisional), with, perhaps, intermediate members. It may be of interest that a preliminary study of silver ingots of Irish provenance suggests a unit of about 8 gm. This same unit is found amongst the Norwegian balance-weights and is the *ertog*, one third of an *øre*.

APPENDIX

'Ring-money' weights *by J A Graham-Campbell*

The following weights (in grams) are those of all complete examples of 'ring-money' from Scotland now in the National Museum of Antiquities of Scotland, and have kindly been provided by the NMAS Research Laboratory (1974/75):

Burray, Orkney

IL 236 : 43·57	IL 245 : 48·60	IL 255 : 21·70
237 : 13·83	246 : 43·87	256 : 44·20
238 : 44·59	247 : 16·62	257 : 22·55
239 : 21·50	248 : 19·03	258 : 22·45
240 : 48·23	249 : 51·74	259 : 32·47
241 : 50·85	250 : 22·50	260 : 15·21
242 : 39·85	251 : 72·29	261 : 30·03
243 : 28·41	253 : 49·46	
244 : 38·70	254 : 23·83	

Dibadail, Lewis

IL 518 : 47·38	IL 519 : 24·51	IL 520 : 19·68

Jarlshof, Shetland

HSA 995a : 20·09

Kirk o' Banks, Caithness

IL 114 : 47·66	IL 116 : 50·53	IL 118 : 50·81
115 : 50·94	117 : 51·04	

Port Glasgow, Renfrewshire

IL 226 : 38·64

Skaill, Orkney

IL 26 : 54·86	IL 38 : 45·26	IL 45 : 69·13
27 : 37·78	39 : 53·38	46 : 45·26
28 : 31·72	40 : 48·55	47 : 34·99
29 : 118·60	41 : 53·13	48 : 68·56
30 : 57·20	42a : 105·50	49 : 48·31
31 : 49·44	42b : 74·13	88 : 41·48
34 : 25·23	43 : 46·97	515 : 51·01
36 : 34·29	44 : 79·69	

Tarbat, Ross-shire

IL 272 : 19·24	IL 274 : 15·01
273 : 25·82	275 : 21·36

The following weights (in grams) are of complete examples of 'ring-money' from the Isle of Man and have kindly been provided by the Manx Museum and the Department of Medieval and Later Antiquities, British Museum (1975/76):

Douglas

Manx Museum 4411 : 50·94   4414 : 66·55
4412 : 47·50   4421 : 49·70
4413 : 22·20
British Museum 95, 8–9, 4 : 93·0
95, 8–9, 5 : 71·5

Kirk Michael 1972/75[1]

Manx Museum 50·30 (largest arm-ring)
41·34 (arm-ring in two pieces)

West Nappin

Manx Museum 4396 : 21·84

Note

1 Since Warner's statistical analysis was completed using the above weights, Mr A M Cubbon informs me that the two Kirk Michael 1972/5 arm-rings have been re-weighed (in the British Museum). The new weights given are 49·3 gm for the largest ring and 40·29 gm for the other (37·82+2·47 gm).

ACKNOWLEDGMENT

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